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Isotropic Kähler structures on Engel 4-manifolds

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Abstract

Examples of isotropic Kähler manifolds (i.e., $\|\nabla J\|^2 = 0$) which are neither complex nor symplectic, and therefore not indefinite Kähler, are constructed.

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1. Introduction

Let (M, g, J) be an indefinite almost Hermitian manifold with almost complex structure J and J-invariant indefinite metric g of signature $(2p, 2q)$, $p, q \geq 0$. Let ∇ denote the Levi Civita connection of g. If $\nabla J = 0$, then J is integrable and the corresponding Kähler form is closed, i.e., (g, J) becomes a Kähler structure. In this case the square norm $\|\nabla J\|^2$ of ∇J vanishes. However, for indefinite almost Hermitian manifolds, its inverse is not always being true. That is, in general, the vanishing of the square norm $\|\nabla J\|^2$ does not always imply the Kähler condition: $\nabla J = 0$. To the authors' knowledge, no example of indefinite almost Hermitian manifold whose almost complex structure satisfies

$$\text{(i) } \nabla J \neq 0, \qquad \text{(ii) } \|\nabla J\|^2 = 0,$$

is known. We will refer to an indefinite almost Hermitian manifold satisfying $\|\nabla J\|^2 = 0$ as *isotropic Kähler* and *indefinite Kähler* if $\nabla J = 0$.

The main purpose of this note is to show the existence of non-Kähler isotropic Kähler manifolds, where the underline manifold of our study is the 4-space $\mathbb{R}^4$ with an Engel dis-

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tribution. (Note that in the Riemannian category, isotropic Kähler manifolds are necessarily Kähler.)

2. Preliminaries

In this section, we collect some basic material to be needed further on. In addition, we would like to point out the link between indefinite almost Hermitian structures and neutral metrics on 4-manifolds, which is the starting motivation of our study.

2.1. Neutral metrics on 4-manifolds

It is observed in [7,8] that the existence conditions for the following three structures on a 4-manifold M are equivalent to each other:

(i) a non-singular field of oriented tangent 2-planes (simply a field of 2-planes);
(ii) an indefinite metric g of signature $(++--)$, with $SO_o(2,2)$ as the structure group of the tangent bundle TM of M;
(iii) a pair (J, J') of an almost complex structure J and an opposite almost complex structure J'.

The equivalence between (i) and (ii) is essentially covered by Steenrod ([9], Section 40.4). That between (i) and (iii) can easily be seen by the isomorphisms [8]:

$$Gr_2(\mathbb{R}^4) = SO(4)/SO(2) \times SO(2) \cong SO(4)/U(2) \times SO(4)/U'(2), \tag{1}$$

where $Gr_2(\mathbb{R}^4)$ is the Grassmann manifold of 2-planes in $\mathbb{R}^4$, and the quotient $SO(4)/U(2)$ (resp. $SO(4)/U'(2)$) is considered as the space of (resp., opposite) almost complex structures on $\mathbb{R}^4$. The identification (1) can be realized explicitly in the following ([3], Proposition 3.1):

Theorem 2.1. *Let (M, h) be a compact oriented Riemannian 4-manifold, endowed with an indefinite metric g of signature $(++--)$. Then, M admits a pair (J, J') of an almost complex structure J and an opposite almost complex structure J', which is uniquely associated with the section Q of $Gr_2(M)$ as follows:*

$$Q = -JJ' = -J'J. \tag{2}$$

2.2. Engel structures

An *Engel structure* on a 4-manifold M is an absolutely non-integrable regular two-dimensional distribution $\mathcal{D} \subset TM$; this means that the Lie bracket $\mathcal{D}^2 = [\mathcal{D}, \mathcal{D}]$ is a three-dimensional distribution and $[\mathcal{D}, \mathcal{D}^2] = TM$. A pair $(M, \mathcal{D})$ is said to be an *Engel manifold* [4,5].

In a neighborhood of every point of an Engel manifold there is a pair of vector fields W, X which span $\mathcal{D}$ and which, together with their Lie brackets $[W, X]$, $[X, [W, X]]$, span the whole tangent space. There is a Darboux theorem for Engel structures: any two are locally diffeomorphic. Any Engel structure has a canonical line subbundle $L \subset V$ defined in terms of Lie brackets. If an oriented manifold admits an oriented Engel structure then it is necessarily parallelizable.

By Engels Theorem ([1], p. 50), the 4-space $\mathbb{R}^4$ can be covered by coordinates (w, x, y, z), such that an Engel distribution $\mathcal{D}$ on $\mathbb{R}^4$ can be spanned by two vector fields

$$\mathcal{D} = \text{span}\left\{ W = \frac{\partial}{\partial w}, X = \frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \right\}.$$

The Lie brackets of these vector fields are as follows:

$$[X, W] = -\frac{\partial}{\partial y}, \quad [X, [X, W]] = \frac{\partial}{\partial z}, \quad [W, [X, W]] = 0.$$

Note that $\mathcal{D} = \text{span}\{X, W\}$ (of dimension 2), $\mathcal{D}^2 = \text{span}\{X, W, [X, W]\}$ (of dimension 3) and $\mathcal{D}^3 = \text{span}\{X, W, [X, W], [X, [X, W]], [W, [X, W]]\}$ (of dimension 4). We can choose a frame $\{W, X, Y, Z\}$ of vectors at each point (w, x, y, z) as

$$W = \frac{\partial}{\partial w}, \quad X = \frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \quad Y = -\frac{\partial}{\partial y}, \quad Z = -\frac{\partial}{\partial z}. \tag{3}$$

Note that non-zero Lie brackets of these vectors are

$$[W, X] = -Y, \qquad [X, Y] = -Z. \tag{4}$$

These four vectors W, X, Y, Z are globally defined on $\mathbb{R}^4$. The dual frame of 1-forms (the normal forms for the Engel distribution) is given by

$$e^W = \mathrm{d}w, \quad e^X = \mathrm{d}x, \quad e^Y = w\,\mathrm{d}x - \mathrm{d}y, \quad e^Z = y\,\mathrm{d}x - \mathrm{d}z.$$

3. Isotropic Kähler manifolds

For a given almost Hermitian manifold (M^{2n}, g, J), the space of linear invariants of order 2 is given by [6]

$$I_n = \{\|\nabla\Omega\|^2, \|\mathrm{d}\Omega\|^2, \|\delta\Omega\|^2, \|N_J\|^2, \tau, \tau^*\},$$

where

$$\|\nabla\Omega\|^2 = \sum_{a,b,c=1}^{2n} \varepsilon_a\varepsilon_b\varepsilon_c(\nabla_{e_a}\Omega)(e_b, e_c)^2, \qquad \|\mathrm{d}\Omega\|^2 = \sum_{a,b,c=1}^{2n} \varepsilon_a\varepsilon_b\varepsilon_c\,\mathrm{d}\Omega(e_a, e_b, e_c)^2,$$

$$\|\delta\Omega\|^2 = \sum_{a=1}^{2n} \varepsilon_a\delta\Omega(e_a)^2, \qquad \|N_J\|^2 = \sum_{a,b,c=1}^{2n} \varepsilon_a\varepsilon_b\|N_J(e_a, e_b)\|^2,$$

$$\tau = \sum_{a,b=1}^{2n} \varepsilon_a\varepsilon_b R(e_a, e_b, e_a, e_b), \qquad \tau^* = \frac{1}{2}\sum_{a,b=1}^{2n} \varepsilon_a\varepsilon_b R(e_a, Je_a, e_b, Je_b).$$

Here $\{e_1, \dots, e_{2n}\}$ is a local orthonormal basis, the curvature tensor is taken with the sign convention $R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, the Kähler 2-form is given by $\Omega(X, Y) = g(JX, Y)$ and the Nijenhuis tensor $N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$.

Further note that if (M, g, J) is four-dimensional, then

$$I_2 = \{\|\nabla\Omega\|^2, \|N_J\|^2, \tau, \tau^*\},$$

and moreover, if we specialize the class of almost Hermitian manifolds, there are some relations among the invariants as follows [6]:

- Kähler manifolds: $I_2 = \langle\{\tau\}\rangle$, since

$$\tau = \tau^*, \qquad \|\nabla\Omega\|^2 = \|d\Omega\|^2 = \|N_J\|^2 = 0. \tag{5}$$

- Almost Kähler manifolds: $I_2 = \langle\{\tau, \tau^*\}\rangle$, since

$$\tau - \tau^* = -\tfrac{1}{2}\|\nabla\Omega\|^2 = -\tfrac{1}{8}\|N_J\|^2. \tag{6}$$

- Hermitian manifolds: $I_2 = \langle\{\|\nabla\Omega\|^2, \tau, \tau^*\}\rangle$, since

$$\|N_J\|^2 = 0. \tag{7}$$

Also note that for an arbitrary almost Hermitian 4-manifold, the following holds

$$\|\delta\Omega\|^2 = \tfrac{1}{6}\|d\Omega\|^2, \qquad \|\nabla\Omega\|^2 = \tfrac{1}{3}\|d\Omega\|^2 + \tfrac{1}{4}\|N_J\|^2. \tag{8}$$

In what remains of this section we will construct two families of isotropic Kähler manifolds (i.e., $\|\nabla J\|^2 = 0$). In both cases we will work on the basis of the Euclidean four-dimensional space and Engel structures as discussed in Section 2.2.

3.1. *Double isotropic Kähler structures which are neither complex nor symplectic*

Naturally associated to an Engel distribution $\mathcal{D}$ on $\mathbb{R}^4$, consider the distributions $\mathcal{D}^+ = \operatorname{span}\{W, X\}$, $\mathcal{D}^- = \operatorname{span}\{Y, Z\}$ and associate a pair of an almost complex structure J and an opposite one J' as follows:

$$JW = X, \quad JX = -W, \quad JY = Z, \quad JZ = -Y,$$
$$J'W = X, \quad J'X = -W, \quad J'Y = -Z, \quad J'Z = Y.$$

Define a pseudo-Riemannian metric g of signature $(++--)$ as follows. Consider the four vector fields W, X, Y, Z, the associated 1-forms and put

$$\begin{aligned} g &= (e^W)^2 + (e^X)^2 - (e^Y)^2 - (e^Z)^2 \\ &= dw^2 + (1 - w^2 - y^2)dx^2 - dy^2 - dz^2 + 2w\,dx\,dy + 2y\,dx\,dz \end{aligned}$$

It is clear that (M, g, J, J') is a double almost Hermitian manifold. Moreover, the non-zero components of the Levi Civita connection ∇ of g are given by

$$\nabla_W X = -\tfrac{1}{2}Y, \quad \nabla_W Y = -\tfrac{1}{2}X, \quad \nabla_X Y = \tfrac{1}{2}W - \tfrac{1}{2}Z,$$
$$\nabla_X Z = \tfrac{1}{2}Y, \quad \nabla_Y Z = \tfrac{1}{2}X. \tag{9}$$

Now, a long but straightforward calculation shows that

$$\begin{aligned}\|\nabla J\|^2 &= \sum_{i,j,k,l} g^{ij} g^{kl} g((\nabla_i J)e_k, (\nabla_j J)e_l) \\ &= 4\left\{-\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right\} = 0.\end{aligned}$$

and

$$\begin{aligned}\|\nabla J'\|^2 &= \sum_{i,j,k,l} g^{ij} g^{kl} g((\nabla_i J)e_k, (\nabla_j J)e_l) \\ &= 4\left\{-\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right\} = 0.\end{aligned}$$

This shows that M is endowed with two kinds of indefinite almost Hermitian structures (g, J) and (g, J') which are *isotropic Kähler but neither complex nor symplectic*.

It follows from (9) that the non-zero components of the curvature tensor R of (M, g) are given by

$$\begin{aligned}&R_{WXWX} = -\tfrac{3}{4}, \quad R_{WXZX} = \tfrac{1}{4}, \quad R_{WYWY} = -\tfrac{1}{4}, \quad R_{WYWZ} = -\tfrac{1}{4}, \\ &R_{YZYZ} = -\tfrac{1}{4}, \quad R_{ZXZX} = \tfrac{1}{4}, \quad R_{XYXY} = -1.\end{aligned} \tag{10}$$

Hence it follows from (4) and (10), that the linear invariants of (g, J) and (g, J') are as follows:

$$(g, J): \begin{cases} \|\nabla\Omega\|^2 = 0, \\ \|\mathrm{d}\Omega\|^2 = -6, \\ \|N_J\|^2 = 8, \\ \tau = 0,\ \tau^* = -2. \end{cases} \qquad (g, J'): \begin{cases} \|\nabla\Omega'\|^2 = 0, \\ \|\mathrm{d}\Omega'\|^2 = -6, \\ \|N_{J'}\|^2 = 8, \\ \tau = 0,\ \tau^{*'} = -2. \end{cases}$$

3.2. Double isotropic Kähler structures with vanishing linear invariants

Let $M = \mathbb{R}^4$ with an Engel distribution $\mathcal{D}$ as discussed in Section 2.2 and consider the two-dimensional distributions $\mathcal{D}_+ = \operatorname{span}\{W, Y\}$ and, $\mathcal{D}_- = \operatorname{span}\{X, Z\}$. Let $\tilde{g}$ be the pseudo-Riemannian metric given by

$$\begin{aligned}\tilde{g} &= (\mathrm{e}^W)^2 - (\mathrm{e}^X)^2 + (\mathrm{e}^Y)^2 - (\mathrm{e}^Z)^2 \\ &= \mathrm{d}w^2 - (1 + y^2 - w^2)\mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 + 2y\,\mathrm{d}x\mathrm{d}z - 2w\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Since $\mathcal{D}_{(\pm)}$ define minimal foliations on $(M, \tilde{g})$, it follows from [2], Theorem 3.6, that M can be equipped with a double almost Kähler structure. Let us define the almost complex structure $\tilde{J}$ and the opposite almost complex structure $\tilde{J}'$ by

$$\begin{aligned}&\tilde{J}(W) = Y, \quad \tilde{J}(Y) = -W, \quad \tilde{J}(X) = Z, \quad \tilde{J}(Z) = -X, \\ &\tilde{J}'(W) = Y, \quad \tilde{J}'(Y) = -W, \quad \tilde{J}'(X) = -Z, \quad \tilde{J}'(Z) = X.\end{aligned}$$

The Kähler forms $\tilde{\Omega}$ and $\tilde{\Omega}'$ associated to the indefinite almost Kähler structure $(\tilde{g}, \tilde{J})$ and the opposite one $(\tilde{g}, \tilde{J}')$ are given by

$$\tilde{\Omega} = (\mathrm{e}^W) \wedge (\mathrm{e}^Y) - (\mathrm{e}^X) \wedge (\mathrm{e}^Z), \qquad \tilde{\Omega}' = (\mathrm{e}^W) \wedge (\mathrm{e}^Y) + (\mathrm{e}^X) \wedge (\mathrm{e}^Z).$$

An easy calculation from (4) shows that the components of the Nijenhuis tensors of $\tilde{J}$ and $\tilde{J}'$ are given by

$$\begin{aligned} &N_{\tilde{J}}(W, X) = -(Y + X), \quad N_{\tilde{J}}(W, Y) = 0, \quad N_{\tilde{J}}(W, Z) = -W + Z, \\ &N_{\tilde{J}}(X, Y) = -Z + W, \quad N_{\tilde{J}}(X, Z) = 0, \quad N_{\tilde{J}}(Y, Z) = X + Y, \end{aligned}$$

and

$$\begin{aligned} &N_{\tilde{J}'}(W, X) = -Y + X, \quad N_{\tilde{J}'}(W, Y) = 0, \quad N_{\tilde{J}'}(W, Z) = W - Z, \\ &N_{\tilde{J}'}(X, Y) = -Z + W, \quad N_{\tilde{J}'}(X, Z) = 0, \quad N_{\tilde{J}'}(Y, Z) = X - Y, \end{aligned}$$

Hence, it follows that $\|N_{\tilde{J}}\|^2 = 0$ and $\|N_{\tilde{J}'}\|^2 = 0$ but clearly neither $\tilde{J}$ nor $\tilde{J}'$ are complex structures. Now, it immediately follows from (6) that $\|\tilde{\nabla}\tilde{J}\|^2 = 0$ and $\|\tilde{\nabla}\tilde{J}'\| = 0$, which shows that $(M, \tilde{g}, \tilde{J})$ and $(M, \tilde{g}, \tilde{J}')$ are *isotropic Kähler almost Kähler manifolds*.

The non-zero components of the Levi Civita connection $\tilde{\nabla}$ of $\tilde{g}$ are given by

$$\begin{aligned} &\tilde{\nabla}_W X = -\tfrac{1}{2}Y, \quad \tilde{\nabla}_W Y = -\tfrac{1}{2}X, \\ &\tilde{\nabla}_X Y = -\tfrac{1}{2}W - \tfrac{1}{2}Z, \quad \tilde{\nabla}_X Z = -\tfrac{1}{2}Y, \quad \tilde{\nabla}_Y Z = -\tfrac{1}{2}X, \end{aligned} \tag{11}$$

and the non-zero components of the curvature tensor $\tilde{R}$ correspond to

$$\begin{aligned} &\tilde{R}_{WXWX} = \tfrac{3}{4}, \quad \tilde{R}_{WXZX} = \tfrac{1}{4}, \quad \tilde{R}_{WYWY} = \tfrac{1}{4}, \quad \tilde{R}_{WYYZ} = -\tfrac{1}{4}, \\ &\tilde{R}_{YZYZ} = \tfrac{1}{4}, \quad \tilde{R}_{ZXZX} = -\tfrac{1}{4}, \quad \tilde{R}_{XYXY} = -1. \end{aligned} \tag{12}$$

Now, a straightforward calculation shows that $\tilde{\tau} = \tilde{\tau}^* = \tilde{\tau}^{*'} = 0$ and thus, all the linear invariants of $(M, \tilde{g}, \tilde{J})$ and $(M, \tilde{g}, \tilde{J}')$ vanish.

Remark. *The isotropic Kähler structures in Sections 3.1 and 3.2 have* $\mathbb{R}^4$ *as the underlying space. However, note that both can be realized as compact manifolds. In order to do that, let* $\mathbf{g}$ *denote the* 3*-step nilpotent Lie algebra with basis* $\{\mathrm{e}^W, \mathrm{e}^X, \mathrm{e}^Y, \mathrm{e}^Z\}$ whose structure equations are given by

$$\mathrm{d}\mathrm{e}^W = \mathrm{d}\mathrm{e}^X = 0, \quad \mathrm{d}\mathrm{e}^Y = \mathrm{e}^W \wedge \mathrm{e}^X, \quad \mathrm{d}\mathrm{e}^Z = \mathrm{e}^X \wedge \mathrm{e}^Y,$$

and let G *denote the associated connected and simply connected nilpotent Lie group. Since the structure constants are rational, it follows from the classical work of Maltsev that* G *possesses a cocompact discrete subgroup* Γ.

Note that the isotropic Kähler structures constructed above can be viewed as left-invariant ones on G and therefore, they descend to the compact nilmanifold $\Gamma \backslash G$.

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